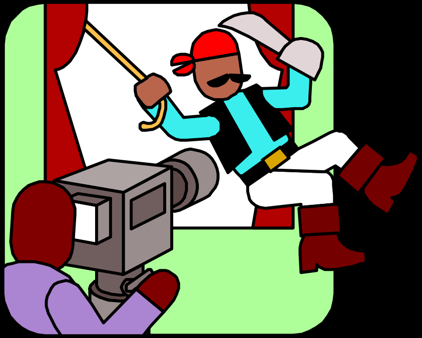
CSC I6716
Spring 2004



Topic 6 of Part 2
Calibration

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http://www-cs.engr.ccny.cuny.edu/~zhu/VisionCourse-2004.html

Lecture Outline

- Calibration: Find the intrinsic and extrinsic parameters
  - Problem and assumptions
  - Direct parameter estimation approach
  - Projection matrix approach

- Direct Parameter Estimation Approach
  - Basic equations (from Lecture 5)
  - Estimating the Image center using vanishing points
  - SVD (Singular Value Decomposition) and Homogeneous System
  - Focal length, Aspect ratio, and extrinsic parameters
  - Discussion: Why not do all the parameters together?

- Projection Matrix Approach (…after-class reading)
  - Estimating the projection matrix M
  - Computing the camera parameters from M
  - Discussion

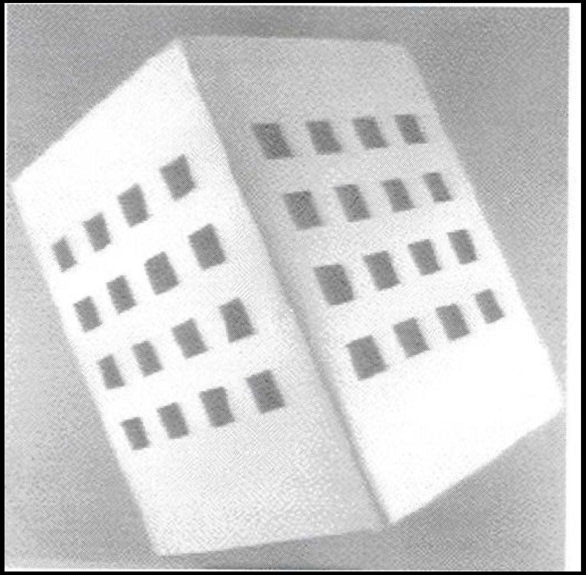
- Comparison and Summary
  - Any difference?

Problem and Assumptions

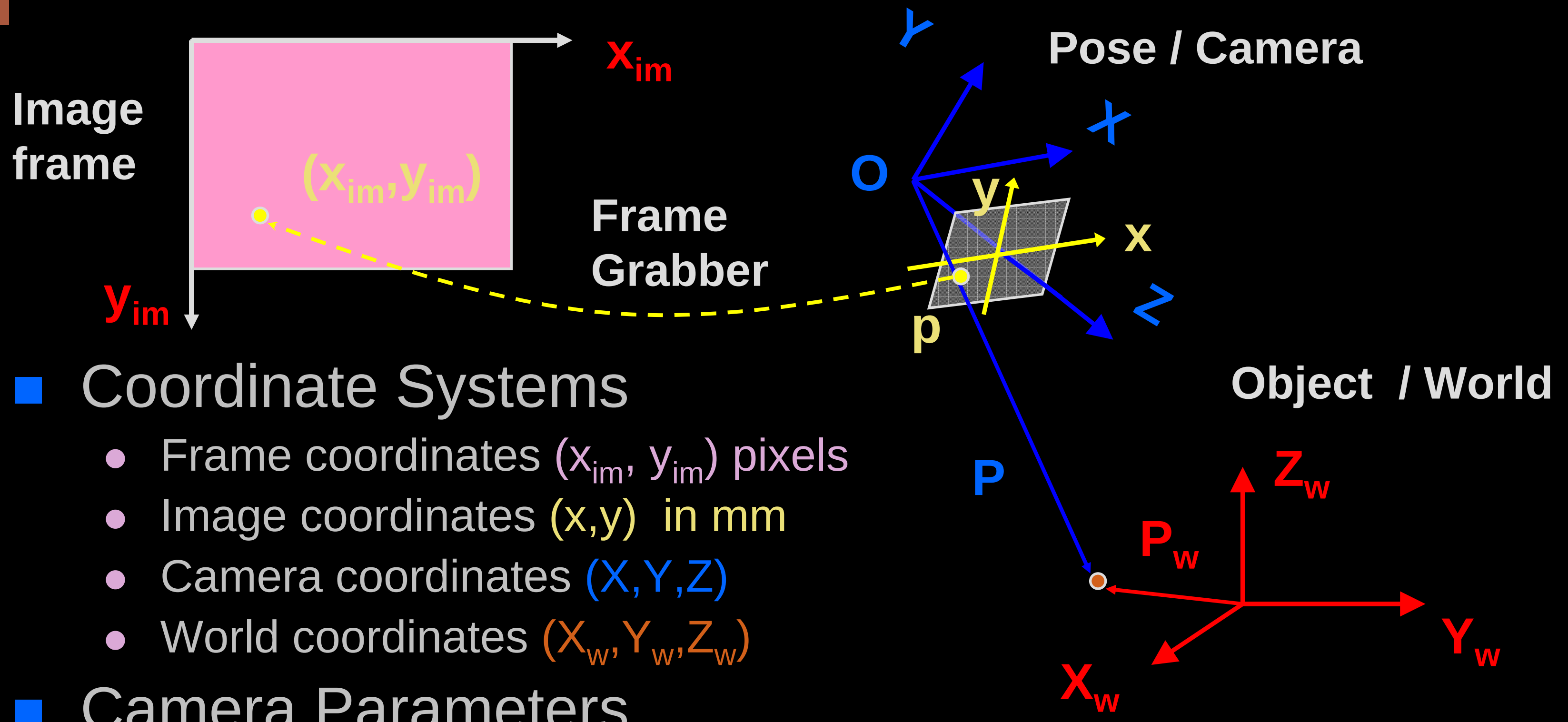
- Given one or more images of a calibration pattern,
- Estimate
  - The intrinsic parameters
  - The extrinsic parameters, or
  - BOTH

- Issues: Accuracy of Calibration
  - How to design and measure the calibration pattern
    - Distribution of the control points to assure stability of solution – not coplanar
    - Construction tolerance one or two order of magnitude smaller than the desired accuracy of calibration
    - e.g. 0.01 mm tolerance versus 0.1mm desired accuracy
  - How to extract the image correspondences
    - Corner detection?
    - Line fitting?
  - Algorithms for camera calibration given both 3D-2D pairs

- Alternative approach: 3D from un-calibrated camera



Camera Model



- Coordinate Systems
  - Frame coordinates (x_{im}, y_{im}) pixels
  - Image coordinates (x,y) in mm
  - Camera coordinates (X,Y,Z)
  - World coordinates (X_w,Y_w,Z_w)
- Camera Parameters
  - Intrinsic Parameters (of the camera and the frame grabber): link the **frame coordinates** of an image point with its corresponding **camera coordinates**
  - Extrinsic parameters: define the location and orientation of the **camera coordinate system** with respect to the **world coordinate system**

Linear Version of Perspective Projection

- World to Camera
  - Camera: P = $(X,Y,Z)^T$
  - World: Pw = (Xw,Yw,Zw)
  - Transform: R, T

$$\mathbf{P} = \mathbf{R}\mathbf{P_w} + \mathbf{T} = \begin{pmatrix} r_{11}X_w + r_{12}Y_w + r_{13}Z_w + T_x \\ r_{21}X_w + r_{22}Y_w + r_{23}Z_w + T_y \\ r_{31}X_w + r_{32}Y_w + r_{33}Z_w + T_z \end{pmatrix} = \begin{bmatrix} \mathbf{R}_1^T\mathbf{P_w} + T_x \\ \mathbf{R}_2^T\mathbf{P_w} + T_y \\ \mathbf{R}_3^T\mathbf{P_w} + T_z \end{bmatrix}$$

- Camera to Image
  - Camera: P = $(X,Y,Z)^T$
  - Image: p = $(x,y)^T$
  - Not linear equations

$$(x, y) = (f\frac{X}{Z},\ f\frac{Y}{Z})$$

- Image to Frame
  - Neglecting distortion
  - Frame $(xim, yim)^T$

$$x = -(x_{im} - o_x)s_x$$
$$y = -(y_{im} - o_y)s_y$$

- World to Frame
  - $(Xw,Yw,Zw)^T$ -> $(xim, yim)^T$
  - Effective focal lengths
    - $f_x = f/s_x$, $f_y=f/s_y$

$$x_{im} - o_x = -f_x \frac{r_{11}X_w + r_{12}Y_w + r_{13}Z_w + T_x}{r_{31}X_w + r_{32}Y_w + r_{33}Z_w + T_z}$$

$$y_{im} - o_y = -f_y \frac{r_{21}X_w + r_{22}Y_w + r_{23}Z_w + T_y}{r_{31}X_w + r_{32}Y_w + r_{33}Z_w + T_z}$$

Direct Parameter Method

$$x' = x_{im} - o_x = -f_x \frac{r_{11}X_w + r_{12}Y_w + r_{13}Z_w + T_x}{r_{31}X_w + r_{32}Y_w + r_{33}Z_w + T_z}$$

$$y' = y_{im} - o_y = -f_y \frac{r_{21}X_w + r_{22}Y_w + r_{23}Z_w + T_y}{r_{31}X_w + r_{32}Y_w + r_{33}Z_w + T_z}$$

- Extrinsic Parameters
  - R, 3x3 rotation matrix
    - Three angles α,β,γ
  - T, 3-D translation vector
- Intrinsic Parameters
  - fx, fy :effective focal length in pixel
    - α = fx/fy = sy/sx, and fx
  - (ox, oy): **known Image center** -> (x,y) known
  - k_1, radial distortion coefficient: **neglect it in the basic algorithm**
- Same Denominator in the two Equations
  - Known : (Xw,Yw,Zw) and its (x,y)
  - Unknown: rpq, Tx, Ty, fx, fy

$$f_y(r_{21}X_w + r_{22}Y_w + r_{23}Z_w + T_y)/y' = f_x(r_{11}X_w + r_{12}Y_w + r_{13}Z_w + T_x)/x'$$

$$x' f_y(r_{21}X_w + r_{22}Y_w + r_{23}Z_w + T_y) = y' f_x(r_{11}X_w + r_{12}Y_w + r_{13}Z_w + T_x)$$

Linear Equations

- Linear Equation of 8 unknowns **v** = (v1,…,v8)
  - Aspect ratio: α = fx/fy
  - Point pairs , {(Xi, Yi,, Zi) <-> (xi, yi) } drop the ‘ and subscript “w”

$$x'(r_{21}X_w + r_{22}Y_w + r_{23}Z_w + T_y) = y'\alpha(r_{11}X_w + r_{12}Y_w + r_{13}Z_w + T_x)$$

$$x_i X_i r_{21} + x_i Y_i r_{22} + x_i Z_i r_{23} + x_i T_y - y_i X_i(\alpha r_{11}) - y_i Y_i(\alpha r_{12}) - y_i Z_i(\alpha r_{13}) - y_i(\alpha T_x) = 0$$

$$x_i X_i v_1 + x_i Y_i v_2 + x_i Z_i v_3 + x_i v_4 - y_i X_i v_5 - y_i Y_i v_6 - y_i Z_i v_7 - y_i v_8 = 0$$

$$(v_1, v_2, v_3, v_4, v_5, v_6, v_7, v_8)$$
$$= (r_{21}, r_{22}, r_{23}, T_y, \alpha r_{11}, \alpha r_{12}, \alpha r_{13}, \alpha T_x)$$

Homogeneous System

- Homogeneous System of N Linear Equations
  - Given N corresponding pairs {(Xi, Yi,, Zi) <-> (xi, yi) }, i=1,2,…N
  - 8 unknowns **v** = (v1,…,v8)^T, **7 independent parameters**

$$x_i X_i v_1 + x_i Y_i v_2 + x_i Z_i v_3 + x_i v_4 - y_i X_i v_5 - y_i Y_i v_6 - y_i Z_i v_7 - y_i v_8 = 0$$

$$\mathbf{Av} = \mathbf{0}$$

$$\mathbf{A} = \begin{bmatrix} x_1 X_1 & x_1 Y_1 & x_1 Z_1 & x_1 & -y_1 X_1 & -y_1 Y_1 & -y_1 Z_1 & -y_1 \\ x_2 X_2 & x_2 Y_2 & x_2 Z_2 & x_2 & -y_2 X_2 & -y_2 Y_2 & -y_2 Z_2 & -y_2 \\ . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . \\ x_N X_N & x_N Y_N & x_N Z_N & x_N & -y_N X_N & -y_N Y_N & -y_N Z_N & -y_N \end{bmatrix}$$

- The system has a nontrivial solution (up to a scale)
  - IF N >= 7 and N points are not coplanar => Rank (**A**) = 7
  - Can be determined from the SVD of A

SVD: definition

Appendix A.6

- Singular Value Decomposition:
  - Any mxn matrix can be written as the product of three matrices

$$\mathbf{A} = \mathbf{UDV}^T$$

$\mathbf{U_1}$ $\mathbf{V_1}$

$$\begin{bmatrix} a_{11} & a_{12} & a_{1n} \\ a_{21} & a_{22} & a_{2n} \\ & & \\ a_{m1} & a_{m2} & a_{mn} \end{bmatrix} = \begin{bmatrix} u_{11} & u_{12} & u_{1m} \\ u_{21} & u_{22} & u_{2m} \\ & & \\ u_{m1} & u_{m2} & u_{mm} \end{bmatrix} \begin{bmatrix} \sigma_1 & 0 & 0 \\ 0 & \sigma_2 & \\ & & \\ 0 & & \sigma_n \\ 0 & & 0 \end{bmatrix} \begin{bmatrix} v_{11} & v_{21} & v_{n1} \\ v_{12} & v_{22} & v_{n2} \\ & & \\ v_{1n} & v_{2n} & v_{nn} \end{bmatrix}$$

- Singular values σ_i are fully determined by A
  - D is diagonal: $d_{ij} = 0$ if $i \neq j$; $d_{ii} = \sigma_i$ (i=1,2,…,n)
  - $\sigma_1 \geq \sigma_2 \geq \ldots \geq \sigma_N \geq 0$
- Both U and V are not unique
  - Columns of each are mutual orthogonal vectors

SVD: properties

$$\mathbf{A} = \mathbf{U}\mathbf{D}\mathbf{V}^T$$

- 1. Singularity and Condition Number
  - nxn A is nonsingular IFF all singular values are nonzero
  - Condition number : degree of singularity of A $C = \sigma_1 / \sigma_n$
    - A is ill-conditioned if 1/C is comparable to the arithmetic precision of your machine; almost singular
- 2. Rank of a square matrix A
  - Rank (A) = number of nonzero singular values
- 3. Inverse of a square Matrix
  - If A is nonsingular $\mathbf{A}^{-1} = \mathbf{V}\mathbf{D}^{-1}\mathbf{U}^T$
  - In general, the pseudo-inverse of A $\mathbf{A}^{+} = \mathbf{V}\mathbf{D}_0^{-1}\mathbf{U}^T$
- 4. Eigenvalues and Eigenvectors (questions)
  - Eigenvalues of both A^TA and AA^T are σ_i^2 ($\sigma_i > 0$)
  - The columns of U are the eigenvectors of AA^T (mxm)
  - The columns of V are the eigenvectors of A^TA (nxn)

$$\mathbf{A}\mathbf{A}^T\mathbf{u}_i = \sigma_i^2\mathbf{u}_i$$

$$\mathbf{A}^T\mathbf{A}\mathbf{v}_i = \sigma_i^2\mathbf{v}_i$$

SVD: Application 1

$$\mathbf{Ax}=\mathbf{b}$$

- Least Square
  - Solve a system of m equations for n unknowns **x**(m >= n)
  - A is a mxn matrix of the coefficients
  - b ($\neq$0) is the m-D vector of the data
  - Solution:

$$\underbrace{\mathbf{A}^T}_{\text{nxn matrix}}\mathbf{Ax}=\mathbf{A}^T\mathbf{b} \Rightarrow \mathbf{x}=\underbrace{(\mathbf{A}^T\mathbf{A})^+}_{\text{Pseudo-inverse}}\mathbf{A}^T\mathbf{b}$$

  - How to solve: compute the pseudo-inverse of A^TA by SVD
    - $(A^TA)^+$ is more likely to coincide with $(A^TA)^{-1}$ given m > n
    - Always a good idea to look at the condition number of A^TA

SVD: Application 2

$$\mathbf{Ax} = \mathbf{0}$$

- Homogeneous System
  - m equations for n unknowns **x**(m >= n-1)
  - Rank (A) = n-1 (by looking at the SVD of A)
  - A non-trivial solution (up to a arbitrary scale) by SVD:
  - Simply proportional to the eigenvector corresponding to the only zero eigenvalue of A^TA (nxn matrix)
- Note:
  - All the other eigenvalues are positive because Rank (A)=n-1
  - For a proof, see Textbook p. 324-325
  - In practice, the eigenvector (i.e. v_n) corresponding to the minimum eigenvalue of A^TA, i.e. σ_n^2

SVD: Application 3

- Problem Statements
  - Numerical estimate of a matrix A whose entries are not independent
  - Errors introduced by noise alter the estimate to Â
- Enforcing Constraints by SVD
  - Take orthogonal matrix A as an example
  - Find the closest matrix to Â, which satisfies the constraints exactly
    - SVD of Â

$$\boxed{\hat{\mathbf{A}} = \mathbf{UDV}^T}$$

    - Observation: D = I (all the singular values are 1) if A is orthogonal
    - Solution: changing the singular values to those expected

$$\boxed{\mathbf{A} = \mathbf{UIV}^T}$$

Homogeneous System

$$\mathbf{A}\mathbf{v} = \mathbf{0}$$

- Homogeneous System of N Linear Equations
  - Given N corresponding pairs {(Xi, Yi,, Zi) <-> (xi, yi) }, i=1,2,…N
  - 8 unknowns **v** = (v1,…,v8)^T, **7 independent parameters**
- The system has a nontrivial solution (up to a scale)
  - IF N >= 7 and N points are not coplanar => Rank (**A**) = 7
  - Can be determined from the SVD of A
  - Rows of V^T: eigenvectors {$\mathbf{e}_i$} of A^{T}A
  - Solution: the 8th row $\mathbf{e}_8$ corresponding to the only zero singular value λ_8=0

$$\mathbf{A} = \mathbf{U}\mathbf{D}\mathbf{V}^T$$

$$\overline{\mathbf{v}} = c\mathbf{e}_8$$

- Practical Consideration
  - The errors in localizing image and world points may make the rank of A to be maximum (8)
  - In this case select the eigenvector corresponding to the smallest eigenvalue.

Scale Factor and Aspect Ratio

- Equations for scale factor γ and aspect ratio α

$$\bar{\mathbf{v}} = \gamma\,(\underset{\mathbf{v}_1}{r_{21}}, \underset{\mathbf{v}_2}{r_{22}}, \underset{\mathbf{v}_3}{r_{23}}, \underset{\mathbf{v}_4}{T_y}, \underset{\mathbf{v}_5}{\alpha r_{11}}, \underset{\mathbf{v}_6}{\alpha r_{12}}, \underset{\mathbf{v}_7}{\alpha r_{13}}, \underset{\mathbf{v}_8}{\alpha T_x})$$

- Knowledge: R is an orthogonal matrix

$$\mathbf{R}_i^T \mathbf{R}_j = \begin{cases} 1 & \text{if} \quad \mathrm{i} = \mathrm{j} \\ 0 & \text{if} \quad \mathrm{i} \neq \mathrm{j} \end{cases}$$

$$\mathbf{R} = (r_{ij})_{3\times3} = \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix} = \begin{bmatrix} \mathbf{R}_1^T \\ \mathbf{R}_2^T \\ \mathbf{R}_3^T \end{bmatrix}$$

- Second row (i=j=2):

$$r_{21}^2 + r_{22}^2 + r_{23}^2 = 1 \Rightarrow |\gamma| = \sqrt{\bar{v}_1^2 + \bar{v}_2^2 + \bar{v}_3^2} \Rightarrow |\gamma|$$

- First row (i=j=1)

$$r_{11}^2 + r_{12}^2 + r_{13}^2 = 1 \Rightarrow \alpha\,|\gamma| = \sqrt{\bar{v}_5^2 + \bar{v}_6^2 + \bar{v}_7^2}$$

$$\left.\begin{array}{l} |\gamma| = \sqrt{\bar{v}_1^2 + \bar{v}_2^2 + \bar{v}_3^2} \\ \alpha\,|\gamma| = \sqrt{\bar{v}_5^2 + \bar{v}_6^2 + \bar{v}_7^2} \end{array}\right\} \alpha$$

Rotation R and Translation T

- Equations for first 2 rows of R and T given α and |γ|

$$\bar{\mathbf{v}} = s\,|\gamma|\,(r_{21}, r_{22}, r_{23}, T_y, \alpha r_{11}, \alpha r_{12}, \alpha r_{13}, \alpha T_x)$$

- First 2 rows of R and T can be found up to a common sign s (+ or -)

$$s\mathbf{R}_1^T, s\mathbf{R}_2^T, sT_x, sT_y$$

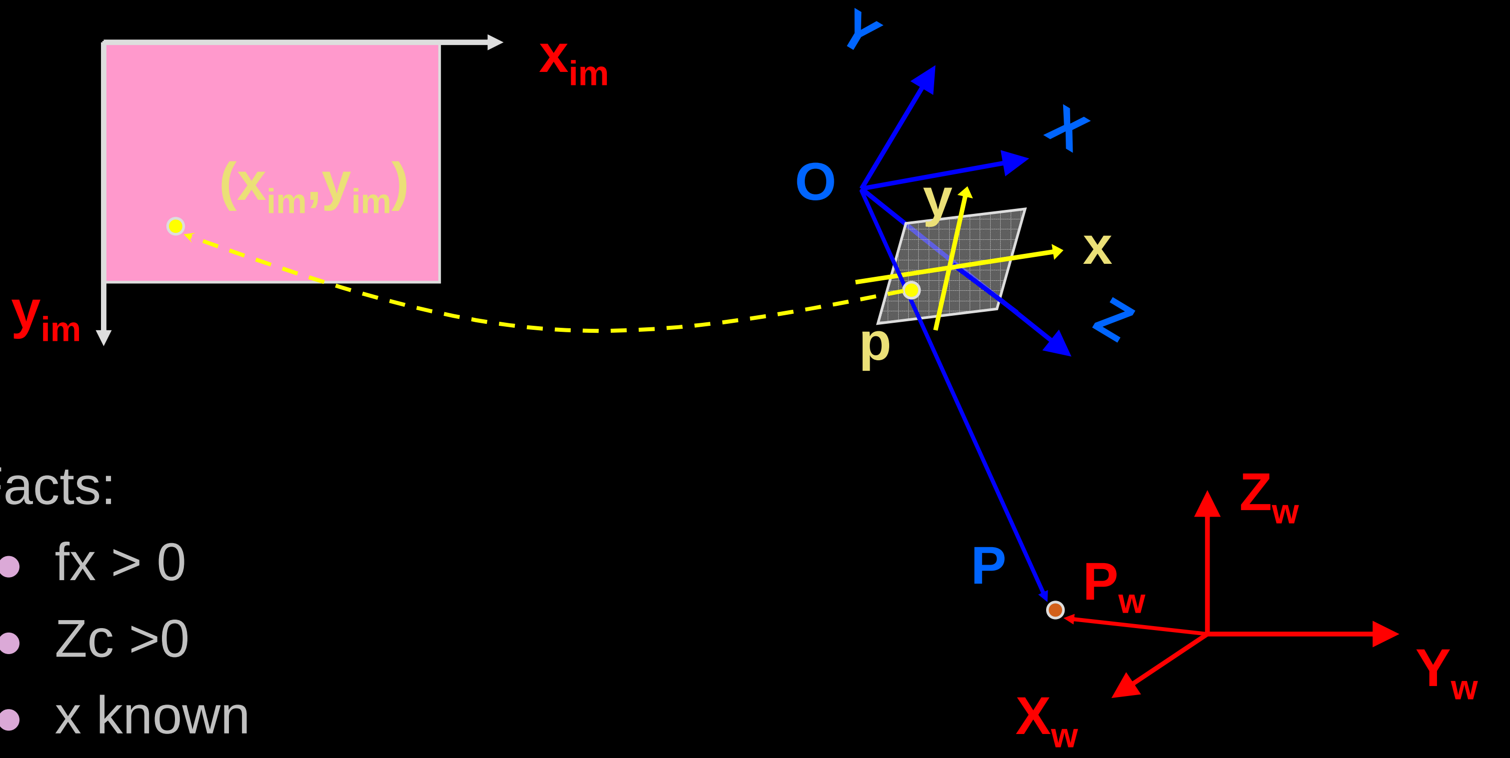
- The third row of the rotation matrix by vector product

$$\mathbf{R}_3^T = \mathbf{R}_1^T \times \mathbf{R}_2^T = s\mathbf{R}_1^T \times s\mathbf{R}_2^T$$

$$\mathbf{R} = \left(r_{ij}\right)_{3\times 3} = \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix} = \begin{bmatrix} \mathbf{R}_1^T \\ \mathbf{R}_2^T \\ \mathbf{R}_3^T \end{bmatrix}$$

- Remaining Questions :
  - **How to find the sign s?**
  - **Is R orthogonal?**
  - **How to find Tz and fx, fy?**

Find the sign s



- Facts:
  - fx > 0
  - Zc >0
  - x known
  - Xw,Yw,Zw known
- Solution
  - ⇒ Check the sign of Xc
  - ⇒ Should be opposite to x

$$x = -f_x \frac{Xc}{Zc} = -f_x \frac{r_{11}X_w + r_{12}Y_w + r_{13}Z_w + T_x}{r_{31}X_w + r_{32}Y_w + r_{33}Z_w + T_z}$$

$$y = -f_y \frac{Yc}{Zc} = -f_y \frac{r_{21}X_w + r_{22}Y_w + r_{23}Z_w + T_y}{r_{31}X_w + r_{32}Y_w + r_{33}Z_w + T_z}$$

Rotation R : Orthogonality

- Question:
  - First 2 rows of R are calculated without using the mutual orthogonal constraint

$$\mathbf{R} = \left(r_{ij}\right)_{3\times3} = \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix} = \begin{bmatrix} \mathbf{R}_1^T \\ \mathbf{R}_2^T \\ \mathbf{R}_3^T \end{bmatrix}$$

$$\mathbf{R}_3^T = \mathbf{R}_1^T \times \mathbf{R}_2^T = s\mathbf{R}_1^T \times s\mathbf{R}_2^T$$

$$\hat{\mathbf{R}}^T \hat{\mathbf{R}} = \mathbf{I}\,?$$

- Solution:
  - Use SVD of estimate R

$$\hat{\mathbf{R}} = \mathbf{U}\mathbf{D}\mathbf{V}^{\mathbf{T}} \quad \Rightarrow \quad \mathbf{R} = \mathbf{U}\mathbf{I}\mathbf{V}^{\mathbf{T}}$$

Replace the diagonal matrix D with the 3x3 identity matrix

Find Tz, Fx and Fy

- Solution
  - Solve the system of N linear equations with two unknown
    - Tx, fx

$$x = -f_x \frac{r_{11}X_w + r_{12}Y_w + r_{13}Z_w + T_x}{r_{31}X_w + r_{32}Y_w + r_{33}Z_w + T_z}$$

$$\underbrace{x}_{\mathbf{a_{i1}}}T_z + \underbrace{(r_{11}X_w + r_{12}Y_w + r_{13}Z_w + T_x)}_{\mathbf{a_{i2}}} f_x = -x\underbrace{(r_{31}X_w + r_{32}Y_w + r_{33}Z_w)}_{\mathbf{bi}}$$

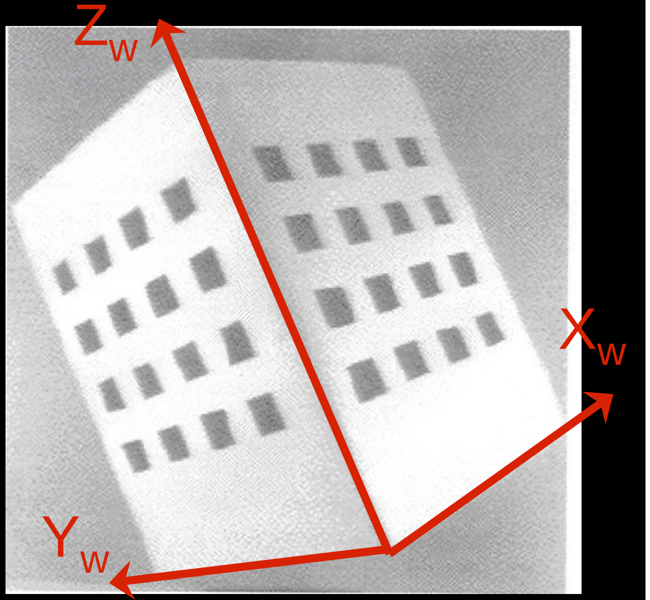
$$\mathbf{A}\begin{pmatrix} T_z \\ f_x \end{pmatrix} = \mathbf{b}$$

  - Least Square method

$$\begin{pmatrix} \hat{T}_z \\ \hat{f}_x \end{pmatrix} = (\mathbf{A}^T\mathbf{A})^{-1}\mathbf{A}^T\mathbf{b}$$

  - SVD method to find inverse

Direct parameter Calibration Summary



- Algorithm (p130-131)

1. Measure N 3D coordinates (Xi, Yi,Zi)
2. Locate their corresponding image points (xi,yi) - Edge, Corner, Hough
3. Build matrix A of a homogeneous system Av = 0
4. Compute SVD of A , solution v
5. Determine aspect ratio α and scale $|\gamma|$
6. Recover the first two rows of R and the first two components of T up to a sign
7. Determine sign s of γ by checking the projection equation
8. Compute the 3rd row of R by vector product, and enforce orthogonality constraint by SVD
9. Solve Tz and fx using Least Square and SVD, then fy = fx / α

Discussions

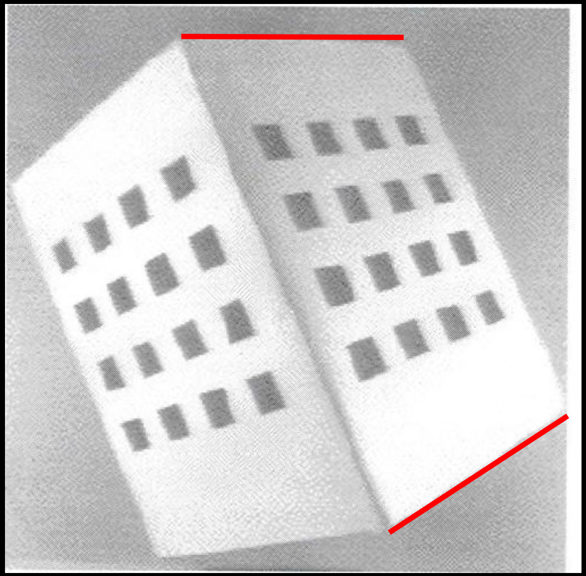
- Questions
  - Can we select an arbitrary image center for solving other parameters?
  - **How to find the image center (ox,oy)?**
  - How about to include the radial distortion?
  - Why not solve all the parameters once ?
    - How many unknown with ox, oy? --- 20 ??? – projection matrix method

$$x = x_{im} - o_x = -f_x \frac{r_{11}X_w + r_{12}Y_w + r_{13}Z_w + T_x}{r_{31}X_w + r_{32}Y_w + r_{33}Z_w + T_z}$$

$$y = y_{im} - o_y = -f_y \frac{r_{21}X_w + r_{22}Y_w + r_{23}Z_w + T_y}{r_{31}X_w + r_{32}Y_w + r_{33}Z_w + T_z}$$

Estimating the Image Center

- Vanishing points:
  - Due to perspective, all parallel lines in 3D space appear to meet in a point on the image - the vanishing point, which is the common intersection of all the image lines



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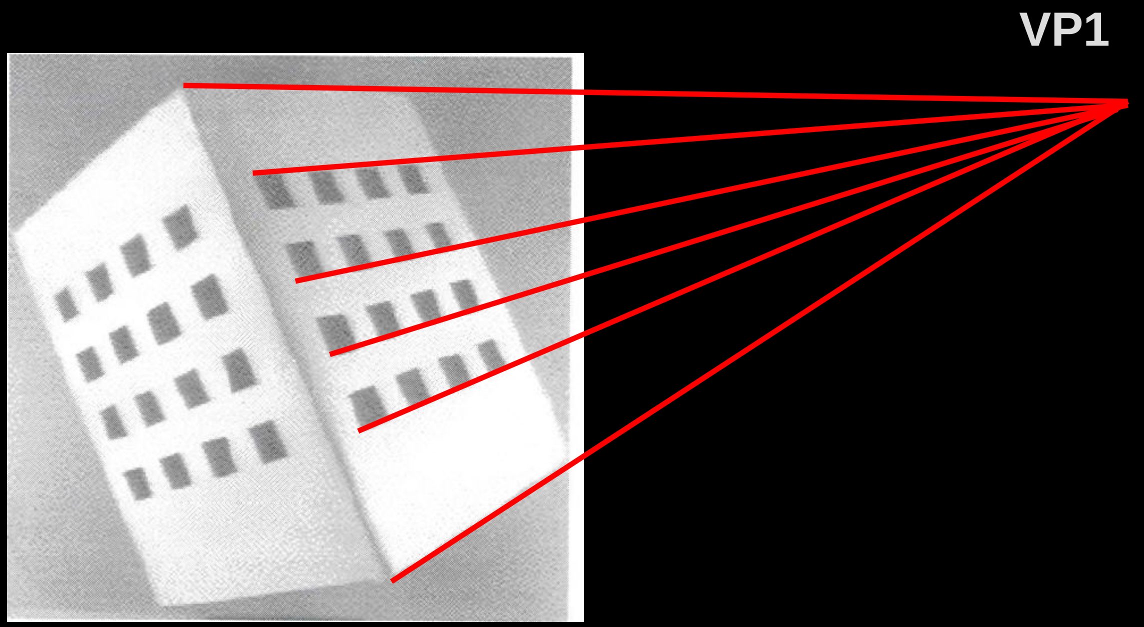
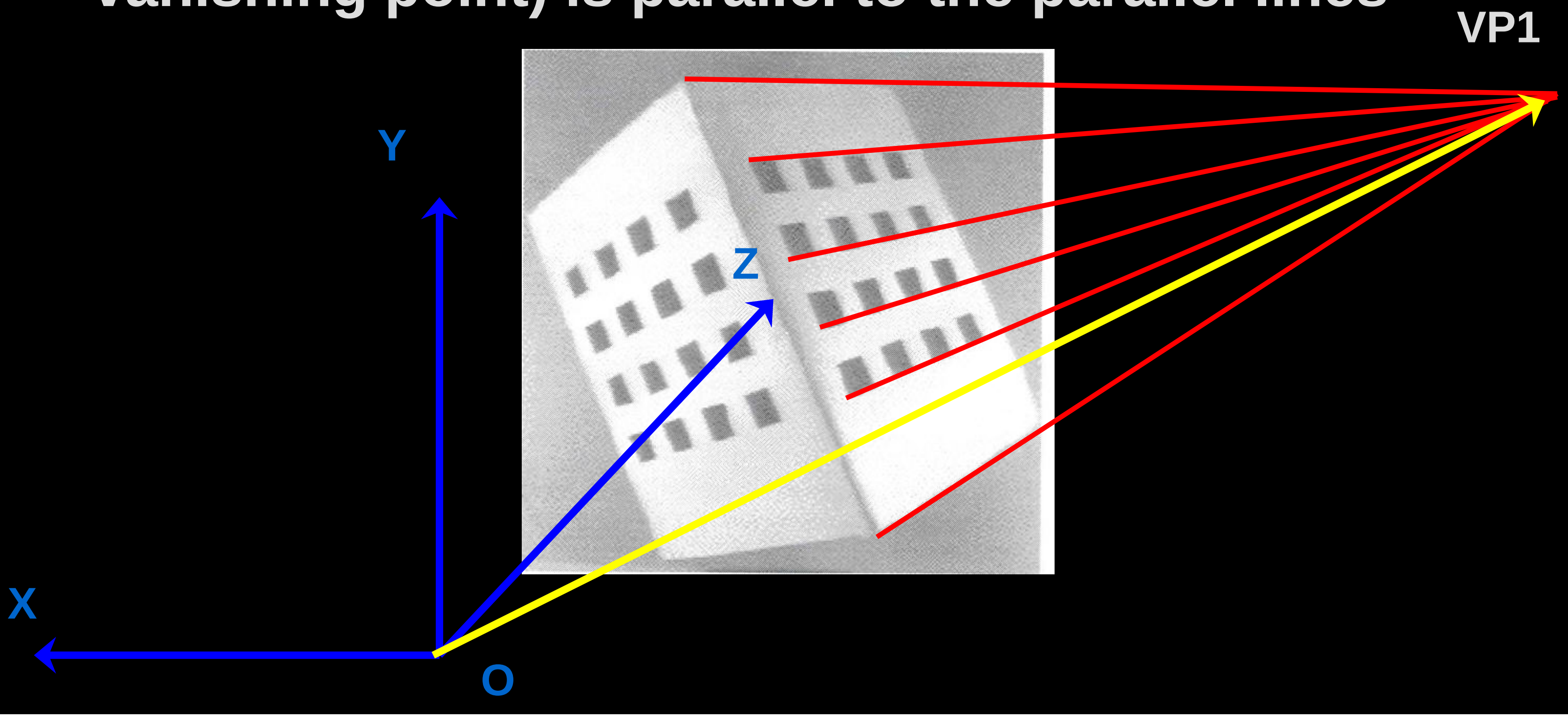


Estimating the Image Center

- Vanishing points:
  - Due to perspective, all parallel lines in 3D space appear to meet in a point on the image - the vanishing point, which is the common intersection of all the image lines
- Important property:
  - **Vector OV (from the center of projection to the vanishing point) is parallel to the parallel lines**



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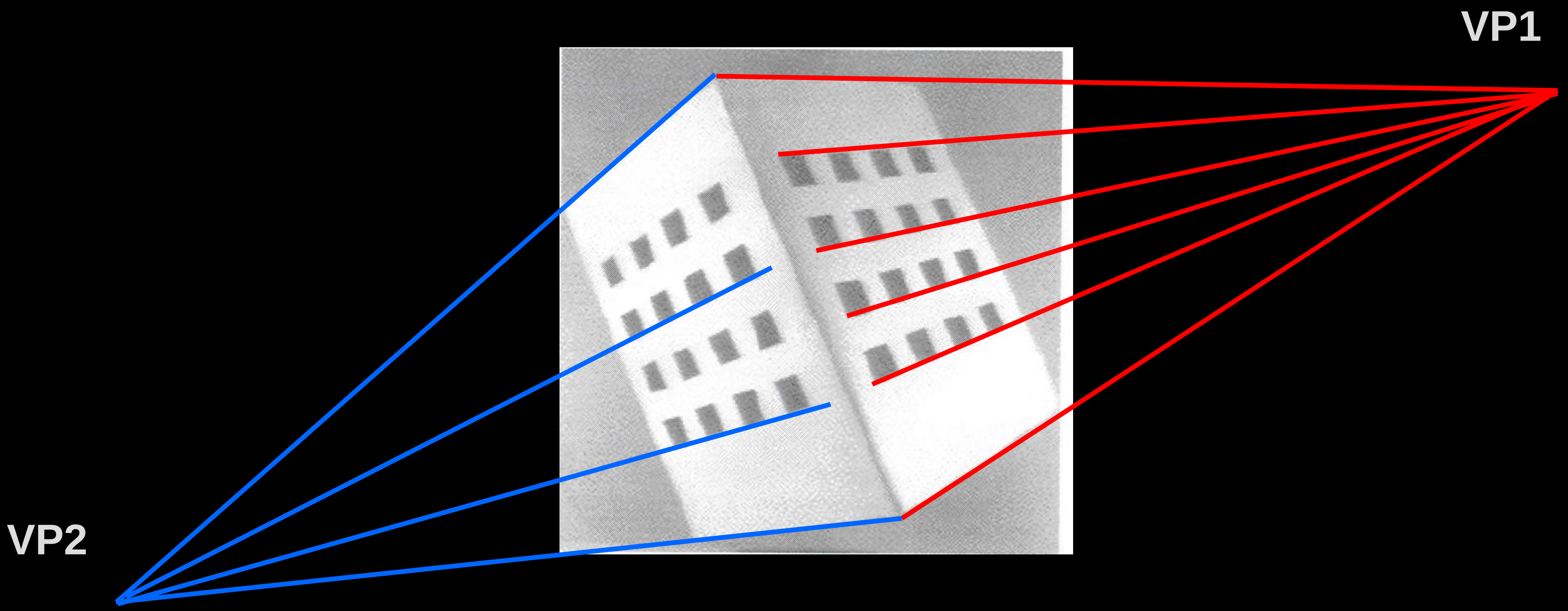
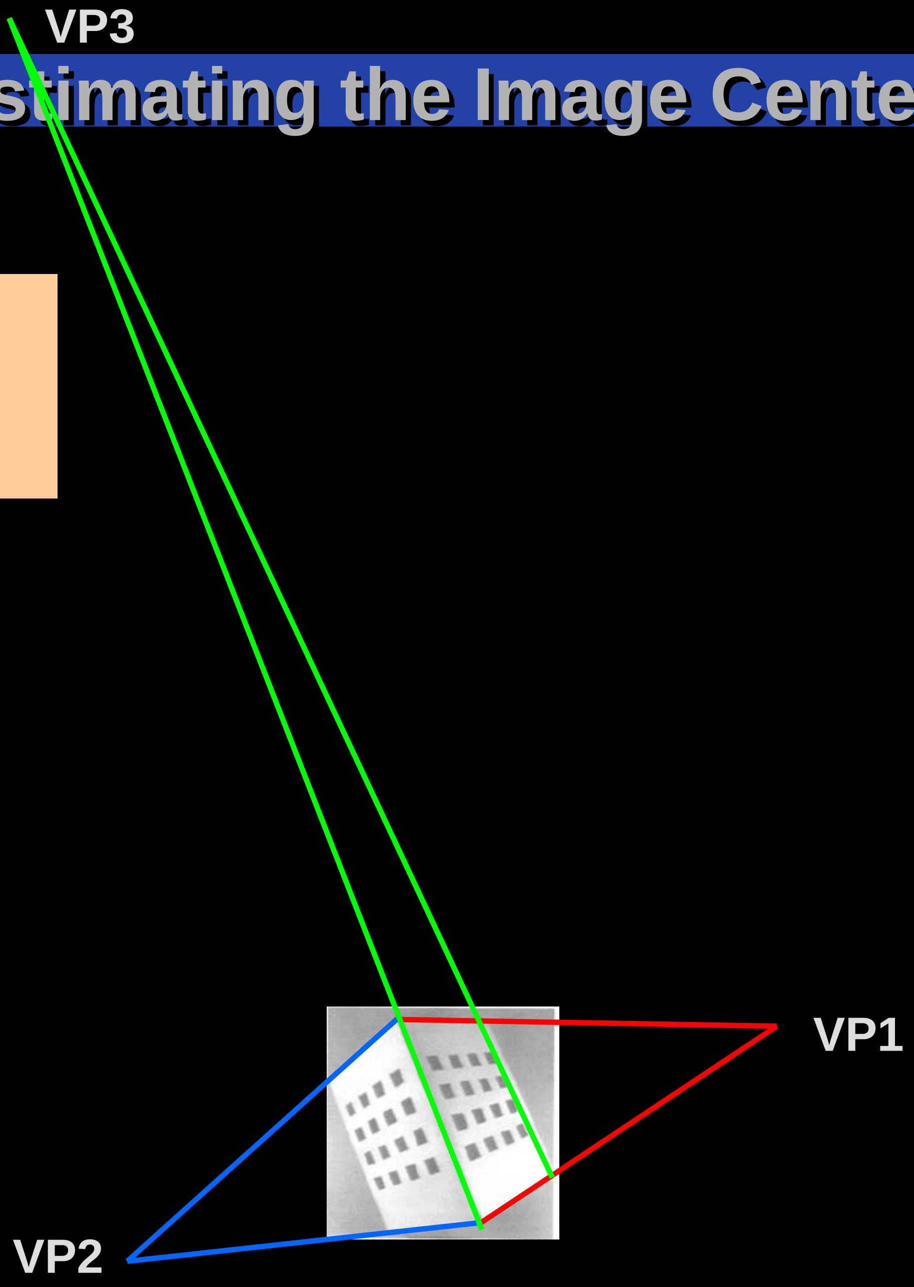


Estimating the Image Center

- Orthocenter Theorem:
  - Input: three mutually orthogonal sets of parallel lines in an image
  - T: a triangle on the image plane defined by the three vanishing points
  - Image center = orthocenter of triangle T
  - Orthocenter of a triangle is the common intersection of the three altitudes



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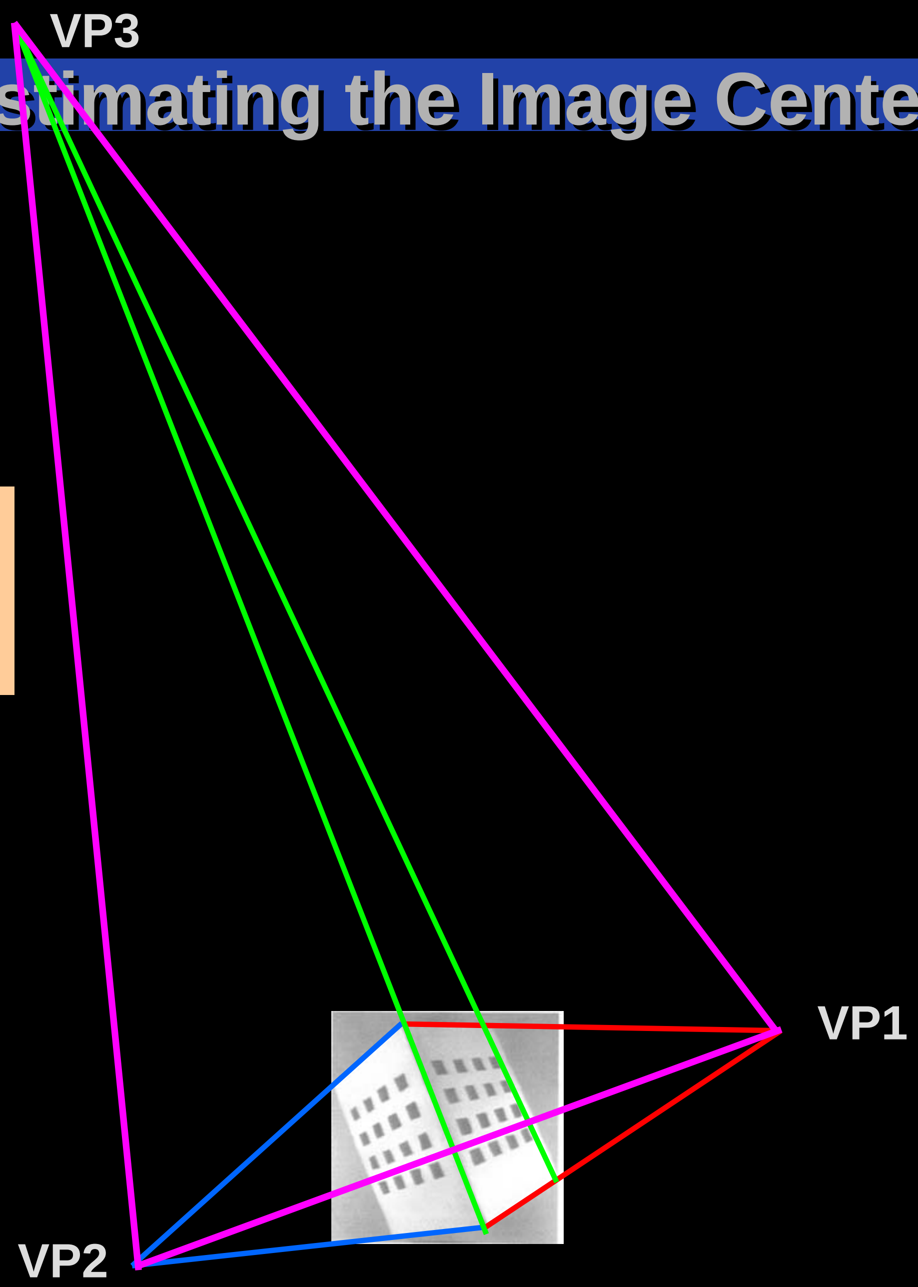

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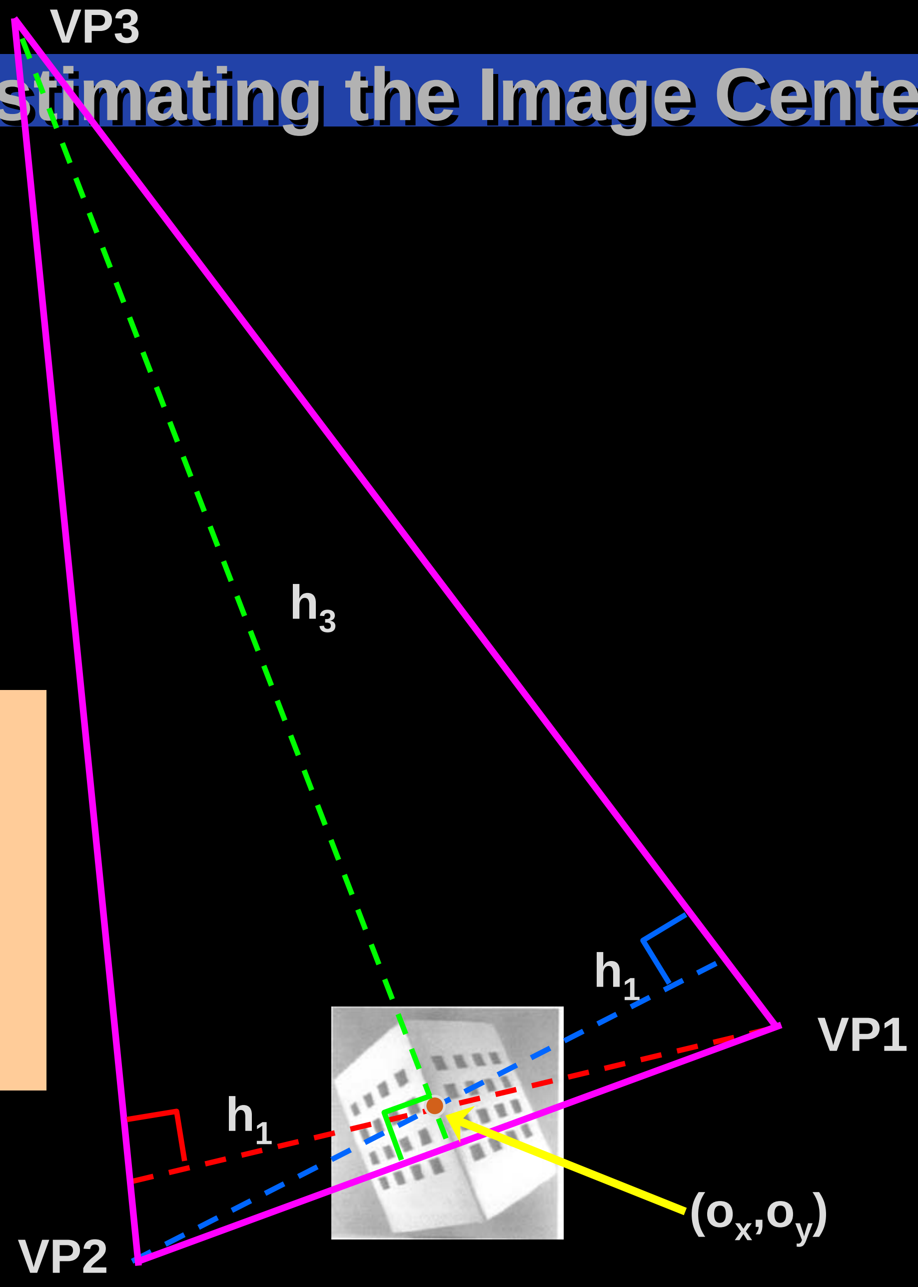
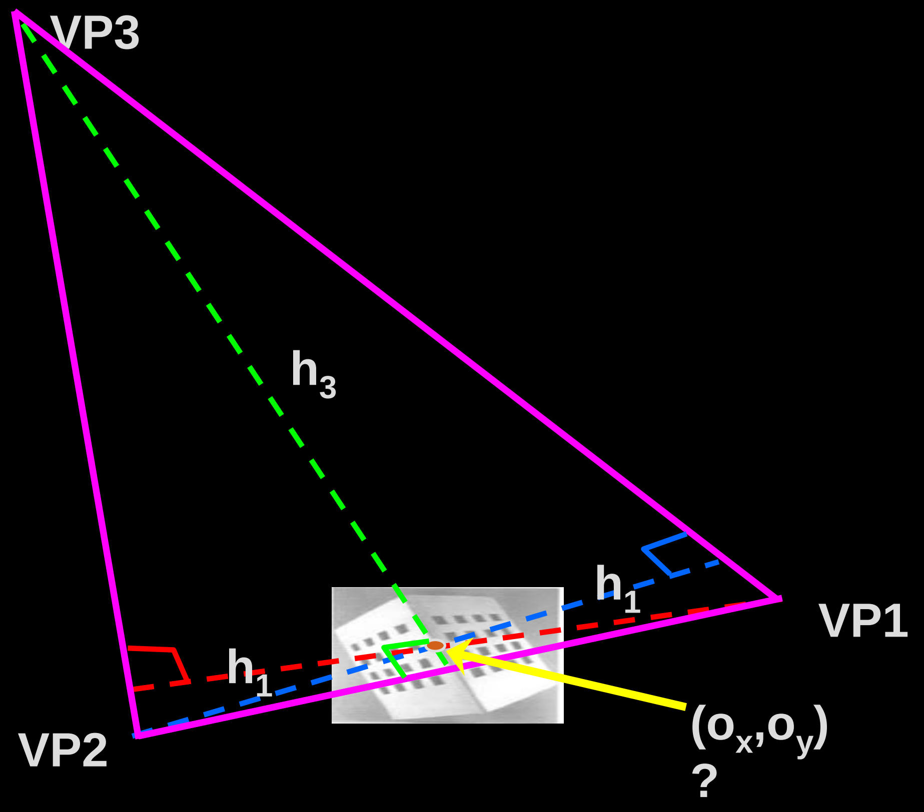
- Orthocenter Theorem:
  - WHY?



Estimating the Image Center

- Assumptions:
  - Known aspect ratio
  - Without lens distortions
- Questions:
  - Can we solve both aspect ratio and the image center?
  - How about with lens distortions?

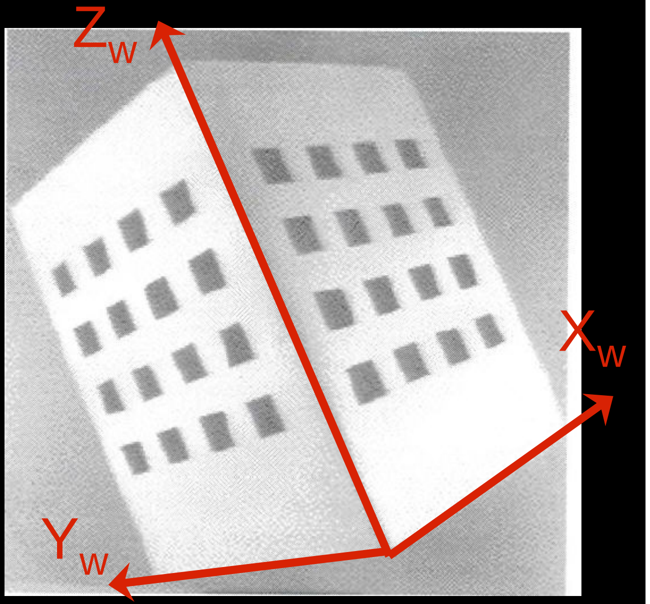


Direct parameter Calibration Summary

- Algorithm (p130-131)

0. Estimate image center (and aspect ratio)

1. Measure N 3D coordinates (Xi, Yi,Zi)
2. Locate their corresponding image (xi,yi) - Edge, Corner, Hough
3. Build matrix A of a homogeneous system Av = 0
4. Compute SVD of A , solution v
5. Determine aspect ratio α and scale $|\gamma|$
6. Recover the first two rows of R and the first two components of T up to a sign
7. Determine sign s of γ by checking the projection equation
8. Compute the 3rd row of R by vector product, and enforce orthogonality constraint by SVD
9. Solve Tz and fx using Least Square and SVD , then fy = fx / α



Remaining Issues and Possible Solution

- Original assumptions:
  - Without lens distortions
  - Known aspect ratio when estimating image center
  - Known image center when estimating others including aspect ratio

- New Assumptions
  - Without lens distortion
  - Aspect ratio is approximately 1, or α = fx/fy = 4:3 ; image center about (M/2, N/2) given a MxN image

- Solution (?)
  1. Using α = 1 to find image center (ox, oy)
  2. Using the estimated center to find others including α
  3. Refine image center using new α ; if change still significant, go to step 2; otherwise stop

Projection Matrix Approach

Linear Matrix Equation of perspective projection

- Projective Space
  - Add fourth coordinate
    - $P_w = (X_w, Y_w, Z_w, 1)^T$
  - Define $(u,v,w)^T$ such that
    - $u/w = x_{im}$, $v/w = y_{im}$
- 3x4 Matrix E_{ext}
  - Only extrinsic parameters
  - World to camera
- 3x3 Matrix E_{int}
  - Only intrinsic parameters
  - Camera to frame

$$\begin{pmatrix} x_{im} \\ y_{im} \end{pmatrix} = \begin{pmatrix} u/w \\ v/w \end{pmatrix} \Leftarrow \begin{pmatrix} u \\ v \\ w \end{pmatrix} = \mathbf{M}_{\mathbf{int}} \mathbf{M}_{\mathbf{ext}} \begin{pmatrix} X_w \\ Y_w \\ Z_w \\ 1 \end{pmatrix}$$

$$\mathbf{M}_{ext} = \begin{bmatrix} r_{11} & r_{12} & r_{13} & T_x \\ r_{21} & r_{22} & r_{23} & T_y \\ r_{31} & r_{32} & r_{33} & T_z \end{bmatrix} = \begin{bmatrix} \mathbf{R}_1^T & T_x \\ \mathbf{R}_2^T & T_y \\ \mathbf{R}_3^T & T_z \end{bmatrix}$$

$$\mathbf{M}_{\mathrm{int}} = \begin{bmatrix} -f_x & 0 & o_x \\ 0 & -f_y & o_y \\ 0 & 0 & 1 \end{bmatrix}$$

- Simple Matrix Product! Projective Matrix $M = M_{int} M_{ext}$
  - $(X_w, Y_w, Z_w)^T$ -> $(x_{im}, y_{im})^T$
  - Linear Transform from projective space to projective plane
  - M defined up to a scale factor – 11 independent entries

Projection Matrix M

- World – Frame Transform
  - Drop “im” and “w”
  - N pairs (xi,yi) <-> (Xi,Yi,Zi)
  - Linear equations of m

$$\begin{pmatrix} u_i \\ v_i \\ w_i \end{pmatrix} = \mathbf{M} \begin{pmatrix} X_i \\ Y_i \\ Z_i \\ 1 \end{pmatrix}$$

$$x_i = \frac{u_i}{w_i} = \frac{m_{11}X_i + m_{12}Y_i + m_{13}Z_i + m_{14}}{m_{31}X_i + m_{32}Y_i + m_{33}Z_i + m_{34}}$$

$$y_i = \frac{u_i}{w_i} = \frac{m_{21}X_i + m_{22}Y_i + m_{23}Z_i + m_{24}}{m_{31}X_i + m_{32}Y_i + m_{33}Z_i + m_{34}}$$

$$\mathbf{Am} = \mathbf{0}$$

- 3x4 Projection Matrix M
  - Both intrinsic (4) and extrinsic (6) – 10 parameters

$$\mathbf{M} = \begin{bmatrix} -f_x r_{11} + o_x r_{31} & -f_x r_{12} + o_x r_{32} & -f_x r_{13} + o_x r_{33} & -f_x T_x + o_x T_z \\ -f_y r_{21} + o_y r_{31} & -f_y r_{22} + o_y r_{32} & -f_y r_{23} + + o_y r_{33} & -f_y T_y + o_y T_z \\ r_{31} & r_{32} & r_{33} & T_z \end{bmatrix}$$

Step 1: Estimation of projection matrix

- World – Frame Transform
  - Drop “im” and “w”
  - N pairs (xi,yi) <-> (Xi,Yi,Zi)

$$x_i = \frac{u_i}{w_i} = \frac{m_{11}X_i + m_{12}Y_i + m_{13}Z_i + m_{14}}{m_{31}X_i + m_{32}Y_i + m_{33}Z_i + m_{34}}$$

$$y_i = \frac{u_i}{w_i} = \frac{m_{21}X_i + m_{22}Y_i + m_{23}Z_i + m_{24}}{m_{31}X_i + m_{32}Y_i + m_{33}Z_i + m_{34}}$$

- Linear equations of m
  - 2N equations, 11 independent variables
  - N >=6 , SVD => m up to a unknown scale

$$\mathbf{Am} = \mathbf{0}$$

$$\mathbf{A} = \begin{bmatrix} X_1 & Y_1 & Z_1 & 1 & 0 & 0 & 0 & 0 & -x_1X_1 & -x_1Y_1 & -x_1Z_1 & -x_1 \\ 0 & 0 & 0 & 0 & X_1 & Y_1 & Z_1 & 1 & -y_1X_1 & -y_1Y_1 & -y_1Y_1 & -y_1 \\ . & . & . & . & . & . & & & & & & \end{bmatrix}$$

$$\mathbf{m} = \begin{bmatrix} m_{11} & m_{12} & m_{13} & m_{14} & m_{21} & m_{22} & m_{23} & m_{24} & m_{31} & m_{32} & m_{33} & m_{34} \end{bmatrix}^T$$

Step 2: Computing camera parameters

$$\hat{\mathbf{M}} = \begin{bmatrix} \mathbf{q}_1 & q_{41} \\ \mathbf{q}_2 & q_{42} \\ \mathbf{q}_3 & q_{43} \end{bmatrix}$$

- 3x4 Projection Matrix M
  - Both intrinsic and extrinsic

$$\mathbf{M} = \begin{bmatrix} -f_x r_{11} + o_x r_{31} & -f_x r_{12} + o_x r_{32} & -f_x r_{13} + o_x r_{33} & -f_x T_x + o_x T_z \\ -f_y r_{21} + o_y r_{31} & -f_y r_{22} + o_y r_{32} & -f_y r_{23} + o_y r_{33} & -f_y T_y + o_y T_z \\ r_{31} & r_{32} & r_{33} & T_z \end{bmatrix}$$

- From M^ to parameters (p134-135)
  - Find scale $|\gamma|$ by using unit vector R_3^T
  - Determine T_z and sign of γ from m_{34} (i.e. q_{43})
  - Obtain R_3^T
  - Find (Ox, Oy) by dot products of Rows q1. q3, q2.q3, using the orthogonal constraints of R
  - Determine fx and fy from q1 and q2 (Eq. 6.19) Wrong???)
  - All the remainings: R_1^T, R_2^T, Tx, Ty
  - Enforce orthognoality on R?

$$\hat{\mathbf{M}} = \gamma \mathbf{M}$$

Comparisons

- Direct parameter method and Projection Matrix method

- Properties in Common:
  - Linear system first, Parameter decomposition second
  - Results should be exactly the same

- Differences
  - Number of variables in homogeneous systems
    - Matrix method: All parameters at once, 2N Equations of 12 variables
    - Direct method in three steps: N Equations of 8 variables, N equations of 2 Variables, Image Center – maybe more stable
  - Assumptions
    - Matrix method: simpler, and more general; sometime projection matrix is sufficient so no need for parameter decompostion
    - Direct method: Assume known image center in the first two steps, and known aspect ratio in estimating image center

Guidelines for Calibration

- Pick up a well-known technique or a few
- Design and construct calibration patterns (with known 3D)
- Make sure what parameters you want to find for your camera
- Run algorithms on ideal simulated data
  - You can either use the data of the real calibration pattern or using computer generated data
  - Define a virtual camera with known intrinsic and extrinsic parameters
  - Generate 2D points from the 3D data using the virtual camera
  - Run algorithms on the 2D-3D data set
- Add noises in the simulated data to test the robustness
- Run algorithms on the real data (images of calibration target)
- If successful, you are all set
- Otherwise:
  - Check how you select the distribution of control points
  - Check the accuracy in 3D and 2D localization
  - Check the robustness of your algorithms again
  - Develop your own algorithms → NEW METHODS?

Next

- 3D reconstruction using two cameras

Stereo Vision

- Homework #3 online, Due April 13 before midnight